Most of the Time it Works Every Time

The Mindset Behind Using Probabilistic Data Structures

Techcamp 2024, Hamburg, Germany
Wolfram „Wolle“ Wingerath
June 19, 2024

Slides Available at https://wolle.science
Skip Lists: Challenge & Basic Idea
Which problem do skip lists solve and in what ways are they superior to other list variants?

Chance, Efficiency & Complexity Analysis
What is the probabilistic element in skip lists, how do they scale, and when should you use them?

Trade-Offs in Other Probabilistic Data Structures
What are advantages of other probabilistic data structures like Bloom filters or Count-Min Sketches?
Helpful Basic Knowledge

Data Structures
Linked Lists, Arrays & Array Lists, Self-Balancing Trees, Hash Maps

Sorted List Applications
Database Sorting & Indexes, Dynamic Collections, (Streaming) Aggregation

Algorithms & Performance Analysis
Binary Search, Tree Traversal, Sorting, Probability Theory basics

Probabilistic Data Structures
Skip Lists & Coin Flipping, Bloom Filters, Count-Min Sketch, Trade-Offs & Use Cases

Learning Goal!

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**Scenario: Working With a Sorted List**

- Imagine **sorted list** of key-value pairs, e.g. ...
  - a sorted set in Redis
  - Member list on your Discord server
  - a list of running medians over a large sliding window

<table>
<thead>
<tr>
<th>keys</th>
<th>values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Benni</td>
<td>3</td>
</tr>
<tr>
<td>Asya</td>
<td>6</td>
</tr>
<tr>
<td>Sarah</td>
<td>9</td>
</tr>
<tr>
<td>Will</td>
<td>12</td>
</tr>
<tr>
<td>Wolle</td>
<td>17</td>
</tr>
<tr>
<td>Fabi</td>
<td>19</td>
</tr>
<tr>
<td>Betsy</td>
<td>21</td>
</tr>
<tr>
<td>Kim</td>
<td>25</td>
</tr>
<tr>
<td>Jane</td>
<td>26</td>
</tr>
</tbody>
</table>

Note: Values will not be visualized on the following slides!
**Challenge**: Maintaining Order

- Why not just use standard list implementations?

Sorted Linked List:  
- Search: $O(n)$
- Update: $O(1)$ (after search)

**VS.**

Sorted Array List:  
- Search: $O(\log n)$
- Update: $O(n)$

Can’t we have a list that gives us both?  
(Yes! Yes, we can!)

Idea for list comparison inspired by:
Skip List **Idea**: A Sorted Linked List Tuned For Binary Search

The perfect skip list is a sorted linked list with **shortcuts** for skipping item subsequences during traversal:

- **Normal Lane** (level 1): standard sorted linked list where every node is connected to its successor
- **Express Lanes** (levels above): Only half of all nodes are promoted to the next level
  - Level 2: add pointers that connect only every 2nd node
  - Level 3: add pointers that connect only every 4th node
  - ... 
  - Level $\log n$: only 1 node that connects to HEAD and TAIL
**Searching the Perfect Skip List**

- **Basic search algorithm:**
  1. Start with the fastest express lane (top level)
  2. Keep advancing until the next step would overshoot, then climb down one level
  3. Repeat until you either find the target or reach the normal lane and find that it’s not in the list

Success!  
× Failure!
### Searching the Perfect Skip List

- **Basic search algorithm:**
  1. Start with the fastest express lane (top level)
  2. Keep advancing until the next step would overshoot, then climb down one level
  3. Repeat until you either find the target or reach the normal lane and find that it’s not in the list

- **$O(\log n)$ Time Complexity:** Search paths no longer than $2 \log n$ nodes
  - There are $\log n$ levels
  - Search will visit no more than 2 nodes per level!

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**Slides:** [https://wolle.science](https://wolle.science)
The Perfect Skip List: Space Efficiency

$O(n)$ Space Complexity: The list has no more than $2n$ pointers

- The number of nodes across all levels can be used as an upper bound:
  - $n$ nodes on level 1 (all nodes)
  - $\frac{n}{2}$ nodes on level 2 (every 2nd node)
  - $\frac{n}{4}$ nodes on level 3 (every 4th node)
  - ...
  - Entire list: $n + \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \cdots = n + n \cdot \sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^k = n + n = 2n$

Geometric series formula:

$$\sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^k = 2$$

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The Perfect Skip List: But What About Updates?

- Value updates are always efficient (search + replace node value)
- Insert and delete operations can be efficient!
  - Example: Inserting 27
    - Structure remains intact with only minor changes (and removing it would be easy as well)
The Perfect Skip List: But What About Updates?

• Value updates are always efficient (search + node value change)
• Insert and delete operations can be efficient!
  o Example: Inserting 27
    ➔ Structure remains intact with only minor changes (and removing it would be easy as well)
• But they can also require (prohibitively) expensive restructuring to keep the perfect structure!
  o Example: Inserting 20
    ➔ Keeping the structure intact is not possible without rearranging many nodes
Probabilistic Structure for Increased Robustness

- **Problem**: efficient updates are not possible while maintaining the perfect skip list structure
- **Approach**: Requirement relaxation!
  - Exactly half of all nodes are promoted to the next level
• **Problem**: efficient updates are not possible while maintaining the perfect skip list structure

• **Approach**: Requirement relaxation!
  
  On average,
  
  ➔ Exactly half of all nodes are promoted to the next level

  ➔ Expected performance remains the same as with perfect skip lists!

• **Coin Flipping**: When inserting a new node, we flip a coin for every promotion decision:
  
  🤔 Heads: The node gets promoted to the next level and we flip again ...

  🤔 Tails: No further promotion!
Deleting From a Skip List

- **Basic delete algorithm** for removing a node X (e.g. 17):

1. Perform search for the to-be-deleted node X until you find the node on the normal lane.

2. On your way down, remember X’s predecessor on every level $\rightarrow$ predecessors = \[
\begin{pmatrix}
12 \\
9 \\
6 \\
6
\end{pmatrix}
\]

3. ...

![Diagram](https://wolle.science)
Deleting From a Skip List

• Basic **delete algorithm** for removing a node $X$ (e.g. 17):

1. Perform search for the to-be-deleted node $X$ until you find the node on the normal lane

2. On your way down, remember $X$’s predecessor on every level → $\text{predecessors} = (6, 9, 12)$

3. Connect $X$’s predecessors with $X$’s successors and remove $X$

---

**delete 17**
Basic **insert algorithm** for adding a node X (e.g. 17) is very similar to the deletion algorithm:

1. Perform search for the to-be-**inserted** node X until you find the position on the normal lane

2. On your way down, remember X’s predecessor on every level \(\rightarrow\) **predecessors** = … (as before)

3. Coin flips to choose a level between 1 and max. level \(\rightarrow\)

4. Insert the node ...

**insert 17**
Inserting Into a Skip List

- Basic **insert algorithm** for adding a node \( X \) (e.g. 17) is very similar to the deletion algorithm:

  1. Perform search for the to-be-inserted node \( X \) until you find the position on the normal lane.

  2. On your way down, remember \( X \)'s predecessor on every level → *predecessors* = ⋯ (as before)

  3. Coin flips to choose a level between 1 and max. level →

  4. Insert the node and update pointers on chosen levels.

**insert 17**
### About Fair & Unfair Coins: Choosing the Optimal p-Value

<table>
<thead>
<tr>
<th>( p )</th>
<th>Time Complexity (Normalized ( \frac{\log_1/p}{p} n ))</th>
<th>Example ( \left( \frac{\log_1/p}{p} n \right) ) for ( n = 128 )</th>
<th>Space Complexity ( \left( \frac{1}{1-p} \right) ) i.e. Avg. Pointers Per Node</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2} = 0.5 )</td>
<td>1</td>
<td>( \frac{\log_2 128}{1/2} = 7 \cdot 2 = 14 )</td>
<td>2</td>
</tr>
<tr>
<td>( \frac{1}{e} \approx 0.368 )</td>
<td>0.942...</td>
<td>( \frac{\log e 128}{1/e} \approx 4.852 \cdot e \approx 13.189 )</td>
<td>1.582...</td>
</tr>
<tr>
<td>( \frac{1}{4} = 0.25 )</td>
<td>1</td>
<td>( \frac{\log_4 128}{1/4} = 3.5 \cdot 4 = 14 )</td>
<td>1.333...</td>
</tr>
<tr>
<td>( \frac{1}{8} = 0.125 )</td>
<td>1.333...</td>
<td>( \frac{\log_8 128}{1/8} \approx 2.333 \cdot 8 \approx 18.666 )</td>
<td>1.143...</td>
</tr>
<tr>
<td>( \frac{1}{16} = 0.0625 )</td>
<td>2</td>
<td>( \frac{\log_{16} 128}{1/16} = 1.75 \cdot 16 = 28 )</td>
<td>1.067...</td>
</tr>
</tbody>
</table>

So decreasing the **p-value** (promotion probability) ...

- ... means **better storage efficiency** (i.e. fewer levels and thus fewer pointers) ...
- ... but also generally **slower searches** (i.e. more steps on avg. search path)!

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Probabilistic Analysis: How Likely is a Slow Search?

But $O(\log n)$ with high probability (w.h.p.) does not give you any strict upper bound, so ...

- ... with some probability, search might still be slow!
- ... in the worst case, a skip list can degrade to a linked list with $\log n$ times the normal pointers!

$\Rightarrow$ Search taking much longer than expected is extremely rare for lists large enough for it to matter!

You buy a lottery ticket and win (6 / 49).

You find a 4-leaf clover on your first try.

Search path is over 3 times longer than expected for a skip list with 4096 items.


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The Skip List: A Probabilistic Alternative to Balanced Trees?

"From a theoretical point of view, there is no need for skip lists. Balanced trees can do everything that can be done with skip lists and have good worst-case time bounds (unlike skip lists)."

— William Pugh (1990)

Both provide $O(\log n)$ time and $O(n)$ space complexity, so why should you choose one over the other?

→ Skip Lists
  - Easy to Build: Simple operations without need for rebalance → typically easier to implement
  - Robustness: performance is unaffected by the order of insertions → no "bad" input sequences

→ Balanced Trees
  - Predictability: Strict worst-case guarantees → no unexpected execution time spikes
  - Efficiency: Constants are often favorable, e.g. high branching factor → shallower structure

Topics for Upcoming Lectures

Advanced Skip List Variations
- Optimizations, Layering Strategies,
- Complexity Analysis

Applications & Benchmarking
- Implementation & Performance Shoot-Out,
- In-Memory vs. Persistent Storage, Tuning

Other Probabilistic Data Structures
- Bloom Filters, Count-Min Sketch, HyperLogLog,
- Trade-Offs & Optimization Goals

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Topics for Upcoming Lectures

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**Bloom Filter Challenge: Checking for Membership**

- **Problem**: Checking the DB for username availability on every registration is expensive!

- **Optimization**: Only ask DB on positive Bloom Filter check!
  - Trade-off: memory efficiency vs. false-positive rate
  - Tuning parameters: number of bits & number of hash functions

- Hash collisions only produce false positives, but never false negatives!

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For every new username, the server ... 
(1) ... computes hash(es) 
(2) ... compares with Bloom Filter 
(3) ... verifies uniqueness just for **positive results** via DB lookup

---

**Hash function**  
(e.g. hash function $h$ that sets a single bit according to the first letter of a name)

| a | b | c | d | e | f | g | h | i | j | k | l | m | n | o | p | q | r | s | t | u | v | w | x | y | z |
| T | T | T | T | T | T | T | T | T | T | T | T | T | T | T | T | T | T | T | T | T | T | T | T | T | T |

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User 
Jane 
Kim 
Betsy 
Fabi 
Wolle 
Will 
Sarah 
Asya 
... 

DB  

Server

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Has username „Zac” been taken already? 
→ **Definitely not!**  
(nobody with starting letter „z”) 

Has username „Wu” been taken already? 
→ **Maybe!**  
(somebody with starting letter „w” → DB lookup)
**Problem**: Checking the DB for username availability on every registration is expensive!

**Optimization**: Only ask DB on positive Bloom Filter check!

- **Trade-off**: memory efficiency vs. false-positive rate

For every new username:
1. Computes hash(es)
2. Compares with Bloom Filter
3. Verifies uniqueness just for positive results via DB lookup

Speed Kit’s web acceleration is only possible because of the Cache Sketch, a probabilistic data structure based on Bloom Filters!
**Challenge: Estimating Item Frequencies**

- **Problem**: The space for keeping one message counter per user grows linearly with your user base!

- **Optimization**: Count hashes instead of users!
  - Trade-off: memory efficiency vs. overcounting error
  - Tuning parameters: number of counters & number of hash functions

- Counts are upper bounds, since hash collisions only lead to overcounting!

---

**Client 2**

- Problem: The space for keeping one message counter per user grows linearly with your user base!

- Optimization: Count hashes instead of users!
  - Trade-off: memory efficiency vs. overcounting error
  - Tuning parameters: number of counters & number of hash functions

- Counts are upper bounds, since hash collisions only lead to overcounting!
Skip Lists combine elements from sorted linked lists and array lists to achieve

- **Simplicity**: straightforward implementation, extension & modification
- **Efficiency**: $O(\log n)$ Time Complexity for inserts, deletes & search with high probability
- **Robustness**: no „bad“ sequences, no rebalancing, no sophisticated tuning required!

**Probabilistic Data Structures** in general are used across a variety of Applications including

- **Order-Preserving Dynamic Collections** (Skip Lists)
- **Efficient Membership Tests Without False Negatives** (Bloom Filters)
- **Estimating Upper Bounds for Item Counts** (Count-Min Sketch)
- **Many More**, e.g. Counting Unique Visitors (HyperLogLog)

**Summing up**: Probabilistic Data Structures Are Awesome!

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